

## UNIT-2 QUANTUM STATISTICS

Statistics!- Collection and analysis of data is known as statistics.

Quantum Statistics!- A statistical description of a system of particles that obey rules of quantum mechanics.

Mechanics!- Science concerned with the motion of bodies under action of force.

Quantum Mechanics!- Deals with smallest thing in universe.

Eg. Electron, Proton, Neutron etc.

Classical Statistics & Quantum Statistics!-

Classical Statistics!- In classical statistics all particles in a system are considered distinguishable. Means individual particle in a system can be tracked. As a result changing the position of any two particles leads to completely different configuration.



( A, B, C are distinguishable particles inside a system )

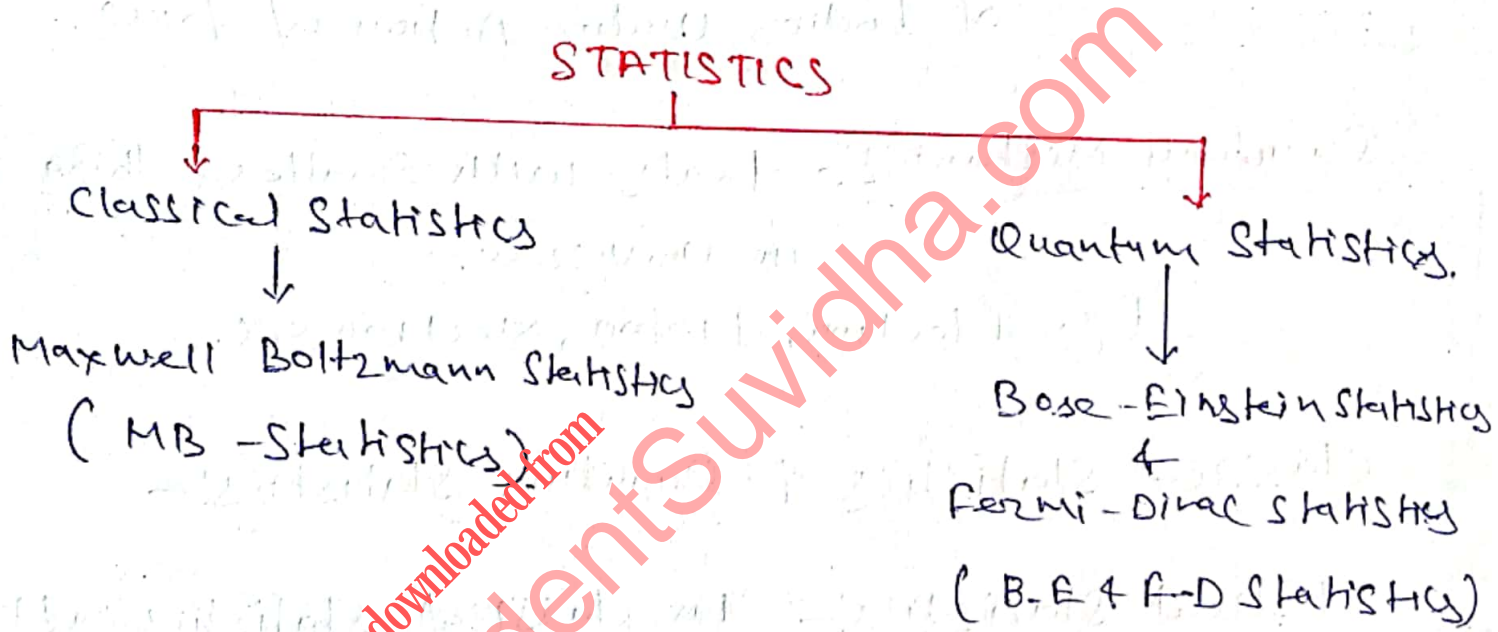
Quantum Statistics:- In quantum statistics, all particles in a system are considered indistinguishable. i.e. we can't label the particle. Interchanging the particle doesn't change the system.

Note:- Apply on Non-observable.

Eg. -



(A, A, A are indistinguishable particles in a system)



### Examples

- I) M-B Statistics:- Gas molecule
- II) B-E Statistics:- photons or Bosons.
- III) F-D Statistics:- Electrons

311 COMPARISON OF MAXWELL-BOLTZMANN, BOSE-EINSTEIN AND FERMI-DIRAC STATISTICS

	Particular	M.B. Statistics	B.E. Statistics	F.D. Statistics
1.	Nature of particles	Particles are identical but distinguishable.	Particles are identical but indistinguishable.	Particles are identical but indistinguishable.
2.	Size of the phase cell	The available volume of the phase space cell can be small as we like and can even approach zero.	Minimum size of the available phase space volume is of the order of $h^3$ , where $h$ is the Planck's constant.	Minimum size of the available phase space volume is of the order of $h^3$ .
3.	Number of phase cells	The phase space can be divided into any large number of cells of our choice.	The phase space can be divided into number of cells whose number is limited as the phase space volume of each cell cannot be less than $h^3$ .	The phase space can be divided into the number of cells, whose number is limited as the phase space volume of each cell cannot be less than $h^3$ .
4.	Number of particles in a cell	Any number of particles can occupy a single phase space cell.	Any number of particles can occupy a single phase cell.	A phase space cell cannot accommodate more than one particle.
5.	Macrostates and microstates	A macrostate can have more than one microstates.	Each macrostate has one microstate.	Each macrostate has one microstate.
6.	Energy	Continuous distribution of energy is assumed.	Energy is quantized.	Energy is quantized.
7.	Spin	The particles are spinless.	The particles have integral spin.	The particles have half integral spin.
8.	Occupation index	$\frac{1}{e^{\alpha + \beta E_i}}$	$\frac{1}{e^{\alpha + \beta E_i} - 1}$	$\frac{1}{e^{\alpha + \beta E_i} + 1}$
9.	Total energy	$\frac{3}{2} Nk_B T$	$\frac{3}{2} Nk_B T \left( 1 - \frac{1}{2^{5/2} e^{-\alpha}} \right)$	$\frac{3}{2} Nk_B T \left( 1 + \frac{1}{2^{5/2} e^{-\alpha}} \right)$
10.	Wave function		Symmetric under interchange of the coordinates of any two Bosons.	Antisymmetric on interchange of the coordinates of any two Fermions.

## Emissive Power:-

Emissive Power of a body at a given temperature is the quantity of Radiant energy emitted by the body per unit time per unit area of the body at that temperature.

## Emissivity ( $\epsilon$ )

Emissivity is defined as the Ratio of Energy Radiated from a material surface to the Radiated Energy of Perfectly black body at that temperature.

## Stefan's Boltzmann law:-

It states that total Radiations emitted from a black body per unit area is proportional to fourth power of its absolute temperature.

( Absolute Temperature - Temperature of Black body )

$$E \propto T^4$$
$$E = \sigma T^4$$

where

- Stefan's constant

$$\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{-K}^4$$

$$E = \frac{Q}{At}$$

$$Q = At \times E$$

$$Q = At \times (\sigma T^4)$$

Q - Amount of heat emitted by a black body.

## Black body Radiation Spectrum:-

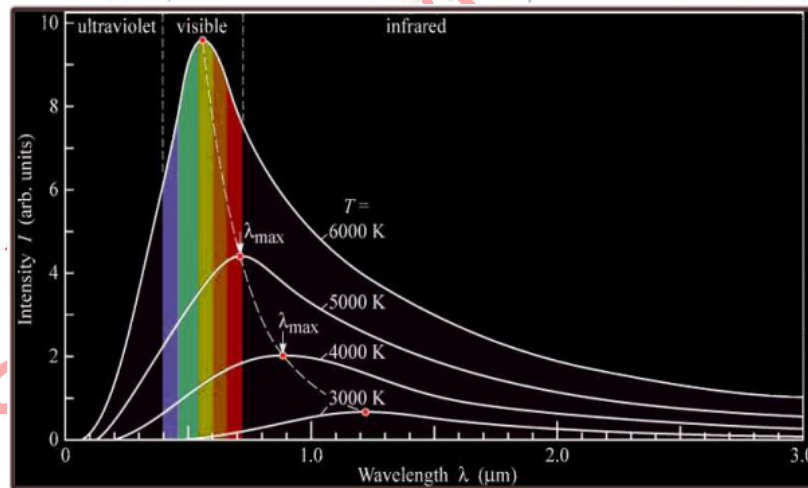
In practice there is not perfect black bodies, Scientist made a blackbody in laboratory by taking hollow sphere & drilling a small hole in it, which can absorb 99% of Radiations.

Inner surface was coated with carbon black when Radiations entering inside hit the wall of sphere and get absorbed 99%.

Remaining radiations reflected back and collide with wall again.

Due to Absorption of heat its temperature inside increase and it starts emitting Radiations from narrow hole. This is called blackbody Radiations.

When radiations from black body traced on a graph classical mechanics fails to explain graph.



- I) The blackbody radiation spectrum is not having uniform distribution of Energy wrt wavelength.
- ii) At a given temperature different wavelength radiations have different energies.
- iii) Total Energy of Radiations at any temperature is given by area under curve. The area is

is directly proportional to fourth power of its absolute temperature  $E \propto T^4$  (Stefan's law)

iv) The wavelength corresponding to maximum energy shift towards shorter wavelength with increase in temperature.  $\lambda_m T = \text{constant}$

(Wien's Displacement law)

$\lambda_m$  - wavelength corresponds to maximum Energy.

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## Wien's Displacement Law:-

Wien's displacement law is the relationship between temperature of a blackbody and wavelength at which it emits light.

Wien's Displacement law states that the blackbody Radiation curve for different temperature peaks at a wavelength is inversely proportional to temperature.

$$\lambda_m \propto \frac{1}{T}$$

$$\lambda_m = \frac{b}{T}$$

b - Wien's constant

$$b = 2.898 \times 10^{-3} \text{ Meter-Kelvin (MK)}$$

## Drawbacks:-

This law is applicable only for short wavelength region & for high temperature of the source of Radiation.

It fails to explain decrease in Energy for longer wavelength.

## Rayleigh Jeans Law:-

Rayleigh Jeans law Express the Energy distribution in the spectrum as a function of Temperature.

$$E_\nu d\nu = \frac{8\pi \nu^2 KT}{c^3} d\nu$$

According to this law Radiation emitted by a particle in a blackbody travel as a wave.

Average Energy per unit vibration =  $KT$

$K$  - Boltzmann constant

$T$  - Temperature.

In term of wavelength Energy density within the range  $\lambda$  to  $\lambda + d\lambda$  is -

Note

$$\left[ \begin{array}{l} \nu = \frac{c}{\lambda} \\ d\nu = -\frac{c}{\lambda^2} d\lambda \\ \text{Since increase in frequency correspond to} \\ \text{decrease in wavelength} \\ E_\nu d\nu = -E_\lambda d\lambda \end{array} \right]$$

inc 1

$$E_\lambda d\lambda = \frac{8\pi \left(\frac{c}{\lambda}\right)^2 KT}{c^3} \times \left(\frac{c}{\lambda^2}\right) d\lambda$$

$$E_\lambda d\lambda = \frac{8\pi KT}{\lambda^4} d\lambda$$

↓

In term of wavelength.



Draw backs:-

Rayleigh Jeans law agree Experimental Results at large wave length (low frequencies). But strongly disagree at shorter wavelength (high frequencies).

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## Planck's law of Radiations

After failure of Wien's law & Rayleigh law  
"Planck's introduced an entirely new idea to explain the distribution of Energy among various wavelength of blackbody Radiation."

Following Assumptions were made -

- I) Blackbody is made up of large number of oscillating particles. These particles vibrate in possible frequencies.
- II) Atoms of the walls of uniform temperature behaves as oscillator each with characteristic frequency of oscillation.
- III) An oscillator can have discrete Energy given by -  
$$E = nh\nu$$

$\nu$  - frequency of oscillator.  
 $h$  - Planck's constant  
 $n$  - an integer quantum number.
- IV) An oscillator does not emit or absorb energy continuously but only in jump.

Let  $N$  be the total number of oscillators &  $E$  be the total Energy of these oscillators, then Average Energy is given by

$$\bar{E} = \frac{E}{N}$$

Now let  $N_0, N_1, N_2, \dots, N_n$  be the number of oscillator having energies  $0, h\nu, 2h\nu, \dots, nh\nu$ .

By Max distribution formula we have

$$N_n = N_0 e^{-nh\nu/KT}$$

According to Maxwell Boltzmann Distribution law -  
 Number of oscillator ( $N_n$ ) with Energy ( $E_n$ ) is given by -

$$N_n = N e^{-E_n/KT}$$

Average Energy of oscillator is given by -

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n N_n}{\sum_{n=0}^{\infty} N_n}$$

$$= \frac{\sum_{n=0}^{\infty} E_n N e^{-E_n/KT}}{N \sum_{n=0}^{\infty} e^{-E_n/KT}}$$

$$= \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/KT}}{e^{-nh\nu/KT}}$$

$$= \frac{0 + h\nu e^{-h\nu/KT} + 2h\nu e^{-2h\nu/KT} + \dots}{1 + e^{-h\nu/KT} + e^{-2h\nu/KT} + \dots}$$

Let  $e^{-h\nu/KT} = x$

$$\Rightarrow \langle E \rangle = \frac{h\nu x (1 + 2x + 3x^2 + \dots)}{1 + x + x^2 + \dots}$$

$$= \frac{h\nu x (1-x)^{-2}}{(1-x)^{-1}}$$

$$= \frac{h\nu x}{(1-x)}$$

$$= \frac{h\nu e^{-h\nu/KT}}{(1 - e^{-h\nu/KT})}$$

$$\boxed{E = \frac{h\nu}{e^{h\nu/KT} - 1}} \quad \text{--- (1)}$$

Average Energy of Planck's oscillation

## Planck's Radiation Formula:-

Number of oscillation in the frequency range  $\nu$  to  $\nu+d\nu$  are estimated to be -

$$\frac{8\pi\nu^2 d\nu}{c^3}$$

The Energy density of Radiations  $E_\nu$  in the frequency range  $\nu$  to  $\nu+d\nu$  is related to average Energy of oscillator as given by.

$$E_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3} \times E$$

$$= \frac{8\pi\nu^2 d\nu}{c^3} \times \frac{h\nu}{e^{h\nu/kT} - 1}$$

(Answer 1)

$$E_\nu d\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu$$

↓  
Planck's Radiation Law.

In term of wavelength:-

Note  $\nu = c/\lambda$   
 $d\nu = -\frac{c}{\lambda^2} d\lambda$   
Since increase in frequency correspond to decrease in wavelength  
 $E_\lambda d\lambda = -E_\nu d\nu$

$$E_\lambda d\lambda = \frac{8\pi h \left(\frac{c}{\lambda}\right)^3 \left(\frac{c}{\lambda^2}\right) d\lambda}{c^3 \times e^{hc/\lambda kT} - 1}$$

$$E_\lambda d\lambda = \frac{8\pi h c}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1}$$

\* Ink  
Wien's law and Rayleigh law from Planck's law!

Wien's law and Rayleigh can be derived from Planck's law:-

From Planck's law we have (by ②)

$$u_{\lambda} d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{e^{hc/\lambda kT} - 1}$$

i) When wave length  $\lambda$  is very small then  $e^{hc/\lambda kT} \gg 1$  so Planck's formula can be (③) written as - (Equation ③)

$$u_{\lambda} d\lambda = \frac{8\pi hc}{\lambda^5} e^{-hc/\lambda kT} d\lambda$$

Putting  $8\pi hc = A$  &  $\frac{hc}{k} = B$  we have

$$u_{\lambda} d\lambda = \frac{A}{\lambda^5} e^{-B/\lambda T} d\lambda$$

↓  
Wien's law.

ii) When  $\lambda$  is very large then -

$$e^{hc/\lambda kT} = 1 + \frac{hc}{\lambda kT}$$

So Equation ③ can be written as -

$$u_{\lambda} d\lambda = \frac{8\pi hc}{\lambda^5 \left(1 + \frac{hc}{\lambda kT} - 1\right)} d\lambda$$

$$u_{\lambda} d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

↓  
Rayleigh's law.

## Photo electric Effect

"When a beam of light of suitable frequency falls on a metal plate slow moving electrons are emitted from the metal surface. This phenomenon is called photoelectric effect"

Some of the basic features of photoelectric effect are -

- I) The lowest frequency of light which can emit photoelectron from a material is called threshold frequency of that material. ( $\nu_0$ )
- II) Photoelectric current or number of photoelectrons emitted per second is directly proportional to the intensity of incident light.
- III) Velocity of photoelectron is completely independent on intensity of incident light but depend on its frequency.
- IV) The Negative potential is at which the photoelectric current is reduced to zero called stopping potential or cut off potential.

When light of frequency  $\nu$  falls on metallic surface a part of its energy is used to liberate the electron from the surface known as work-function and rest energy is carried by emitted electron as kinetic energy.

$$h\nu = \frac{1}{2}mv^2 + \phi_0$$

Kinetic Energy                      work Function.

$$h\nu = \frac{1}{2}mv^2 + h\nu_0$$

( $\nu_0$  - Threshold Frequency)

$$\boxed{\frac{1}{2}mv^2 = h(\nu - \nu_0)}$$
 - Einstein photoelectric Equation.

## Concept of Statistical Mechanics:

Microstate: A state of a system where all the parameters of a system particle are specified.

Many microstates exist for each of the systems, and there are many parameters for each state.

Macrostate: Any state of a system that describes the actual observation of its microscopic statistical properties.

Phase Space: Phase space refers to the plotting of both particle momentum & position on a two-dimensional graph.

If we want to define the state of a particle in phase space coordinate system, then we require position (3D) & momentum (3D). Thus phase space is a six-dimensional coordinate consisting of 3 dimensions of position and 3 dimensions of momentum.

Position Space: Three-dimensional space in which position (location) of a particle is specified. Eg. small volume element ( $dV$ ) in position space is given by -  $dV = dx dy dz$

Momentum Space: Three-dimensional space in which momentum of particle is specified by momentum coordinates  $P_x, P_y, P_z$ .

Cells in Phase Space: - cell in phase space corresponding to a state of a system, can be labeled with some number which.  
A cell is a sub-compartment. Each compartment is divided into large number of cells, in such a way that each cell is of same size.

Has Molecules! Identical particles of any spins & can be distinguished.

Bosons! - Identical particles of 0 or integral spins & can't be distinguished.

Fermions! - Identical particles of odd half integral spins & can't be distinguished.

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## Failure of classical statistics (M-B statistics)

classical statistics fails to explain -

- I) observed energy distribution of Electron in a metal.
- II) observed Energy distribution of photons inside an Enclosure.

That's why we need Quantum Statistics

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## Classical Statistics (Maxwell - Boltzmann Statistics)

"Maxwell Boltzmann law gives distribution of total Energy of a system having identical but distinguishable particles." It is mostly used in kinetic theory of gases to find various aspects such as molecular velocity, probable speed of molecule of gas etc.

### Assumption of M-B Statistics.

- I) All the particles which obey M-B statistics are known as Maxons & are identical & distinguishable
- II) chosen volume has large number of cells and volume of each phase space is extremely small.
- III) There is no limit on number of particles in a cell.
- IV) Total number of particles and total energy remains constant in the system.
- V) The state of each particle is specified by position and momentum coordinates.
- VI) Energy is continuous.

probability  $f(E)$  that a boson occupies a state of energy  $E$  is given by -

$$f(E) = A \exp^{-E/KT}$$

## Maxwell Boltzmann Distribution Law:-

"Maxwell Boltzmann distribution law gives how the total energy of a system is distributed among various member of a system having identical distinguishable particle"

Let us consider an assembly of  $N$  molecule in phase space. Let us divide phase space into number of cells  $1, 2, \dots, k$ , which contain number of particle  $n_1, n_2, \dots, n_k$  respectively in equilibrium. Also suppose energy of  $N$  molecule are limited to  $E_1, E_2, \dots, E_k$  within sequence of energy level.

Note (  $N$  - Molecule  
 $1, 2, 3, \dots, k$  - Number of cells.  
 $n_1, n_2, \dots, n_k$  - Number of particle in cell )

As the molecule are moving continuously in many different ways and obey fundamental postulates of Statistical Mechanics as -

① Total Number of Molecule is constant

$$N = n_1 + n_2 + \dots + n_k = \text{const.}$$

$$\delta N = \delta n_1 + \delta n_2 + \dots + \delta n_k = 0$$

$$\sum \delta n_i = 0 \quad \text{--- (1)}$$

② Total Energy of System is constant.

$$E = E_1 n_1 + E_2 n_2 + \dots + E_k n_k = \text{const.}$$

# ~~Basic System Statistics~~

$$\delta E = E_1 \delta n_1 + E_2 \delta n_2 \dots \delta E_k \delta n_k = 0$$

$$\sum_i E_i \delta n_i = 0 \quad \text{--- (2)}$$

③ When a gas is in Equilibrium probability is maximum.

$$\delta P = 0$$

$$\delta \log P = 0 \quad \text{--- (3)}$$

Now considering

$j_i$  - Number of Energy level each having Energy  $E_i$

$n_i$  - Number of Particles having Energy  $E_i$

Then Probability  $P$  for distribution is given by -

$$P = \frac{N!}{n_1! n_2! n_3! \dots n_k!} (g_1)^{n_1} (g_2)^{n_2} \dots (g_k)^{n_k}$$

$$P = \frac{N!}{n_1! n_2! n_3! \dots n_k!} (g_1)^{n_1} (g_2)^{n_2} \dots (g_k)^{n_k}$$

$$P = \frac{\sum_{i=1}^k (g_i)^{n_i} N!}{\sum_{i=1}^k n_i!}$$

Taking log on both side

$$\log P = \log N! + \sum_i \log (g_i)^{n_i} - \sum_i \log n_i!$$

$$\log P = \log N! + \sum_i n_i \log g_i - \sum_i \log n_i!$$

Using Stirling Approximation  
 $\log N! = N \log N - N$

$$\Rightarrow \log P = N \log N - N + \sum_i n_i \log g_i - \sum_i n_i \log n_i - \sum_i n_i$$

(Using  $\sum_i n_i = N$  (total number of particle)  
 $\sum_i n_i^2 = N$ )

$$= N \log N - N + \sum_i n_i \log g_i - \sum_i n_i \log n_i + N$$

$$\log P = N \log N - \sum_i n_i \log n_i + \sum_i n_i \log g_i$$

(Now Differentiating both side  
 and considering  $N$  &  $g_i$  as constant)

$$\delta \log P = 0 - \left( \sum_i n_i \delta(\log n_i) + \sum_i \log n_i \delta n_i \right) + \sum_i \log g_i \delta n_i$$

$$\delta \log P = 0 - \sum_i \frac{n_i}{g_i} \frac{1}{n_i} \delta n_i - \sum_i \log n_i \delta n_i + \sum_i \log g_i \delta n_i = 0$$

$$= 0 - \alpha - \sum_i \log n_i \delta n_i + \sum_i \log g_i \delta n_i$$

(Using  $\sum_i \delta n_i = \text{Total Number of Molecules} = 0$ )

$$= - \left( \sum_i \log \frac{n_i}{g_i} \delta n_i \right)$$

(Now substituting  $\delta \log P = \text{Total Probability} = 0$ )

$$\Rightarrow \sum_i \log \frac{n_i}{g_i} \delta n_i = 0 \quad \text{--- (4)}$$

Multiplying (1) by  $-\alpha$  (2) by  $-\beta$  and adding (1), (2), (4)

$$\sum_i \log \frac{n_i}{g_i} \delta n_i + \sum_i \alpha \delta n_i + \sum_i \beta \delta n_i = 0$$

$$\sum_i \left( \log \frac{n_i}{g_i} + \alpha + \beta \epsilon_i \right) \delta n_i = 0$$

$$\Rightarrow \log \frac{n_i}{g_i} + \alpha + \beta \epsilon_i = 0$$

$$\log \frac{n_i}{g_i} = -\alpha - \beta \epsilon_i$$

$$\frac{n_i}{g_i} = e^{-\alpha - \beta \epsilon_i}$$

$$n_i = \frac{g_i}{e^{\alpha + \beta \epsilon_i}} = g_i e^{-\alpha} e^{-\beta \epsilon_i}$$

Letting  $A = e^{-\alpha}$

$$n_i = g_i A e^{-\beta \epsilon_i}$$

## Bose-Einstein Statistics:-

"Bose-Einstein Statistics (B.E Statistics) determine the statistical distribution of identical indistinguishable particles (Boson) over the Energy state"

### BASIC Postulates:-

- I) All the particles which obey B.E Statistics are known as Bosons. Bosons are identical & indistinguishable particles.
- II) Bosons obey Uncertainty Principle
- III) Size of cell can't be less than  $h^3$ , where  $h$  is the Planck's constant
- IV) The wave function representing boson do overlap slightly and are symmetric.
- V) The bosons don't obey Pauli Exclusion Principle
- VI) Energy states are discrete.
- VII) Number of phase cells is comparable with number of particles.
- VIII) Bose-Einstein Statistics is applicable to number of particle with integral spin angular momentum in unit of  $(\frac{h}{2\pi})$

Let us consider a system consisting of  $N$  independent and indistinguishable particles. Let there be  $n_1, n_2, n_3, \dots, n_k$  particles, having mean energy  $E_1, E_2, \dots, E_k$  respectively, in compartment numbered as  $1, 2, 3, \dots, k$ , containing  $g_1, g_2, \dots, g_i, \dots, g_k$  cells respectively.

Note  $\left\{ \begin{array}{l} N - \text{Total Number of particles} \\ n_1, n_2, n_3, \dots, n_k - \text{Number of particles in a cell} \\ g_1, g_2, g_3, \dots, g_i, \dots, g_k - \text{cells.} \end{array} \right\}$

Now these particles obey fundamental postulates of quantum statistics as-

① Total Number of particles is constant

$$N = n_1 + n_2 + \dots + n_k = \text{constant}$$

$$\delta N = \delta n_1 + \delta n_2 + \dots + \delta n_k = \text{constant} = 0$$

$$\Rightarrow \boxed{\sum \delta n_i = 0} \quad \text{--- (1)}$$

② Total Energy of system is constant

$$E = E_1 n_1 + E_2 n_2 + \dots + E_k n_k = \text{constant}$$

$$\delta E = E_1 \delta n_1 + E_2 \delta n_2 + \dots + E_k \delta n_k = 0$$

$$\Rightarrow \boxed{\sum E_i \delta n_i = 0} \quad \text{--- (2)}$$

③ Distribution of particles is constant. (Thermodynamic Probability)

$$\frac{\partial \ln W_{20}}{\partial \log W_{20}} \quad \text{--- (3)}$$



Hence the Number of ways in which  $n_i$  Particles are to be distributed in  $g_i$  cells in  $i$ th compartment is given by -

$$w_i = \frac{(n_i + g_i - 1)!}{n_i! (g_i - 1)!}$$

Total Number of Arrangements is -

$$W = \sum_{i=1}^K \frac{(n_i + g_i - 1)!}{n_i! (g_i - 1)!} = \sum_{i=1}^K \frac{(n_i + g_i)!}{n_i! g_i!}$$

Taking log on both sides.

$$\log W = \sum_{i=1}^K (\log (n_i + g_i)! - \log n_i! - \log g_i!)$$

( Using Stirling Approximation.  
 $\log N! = N \log N - N$  )

$$\log W \approx \sum_{i=1}^K ( (n_i + g_i) \log (n_i + g_i) - (n_i + g_i) - n_i \log n_i - n_i - g_i \log g_i - g_i )$$

$$\log W \approx \sum_{i=1}^K ( n_i \log (n_i + g_i) + g_i \log (n_i + g_i) - n_i \log n_i - g_i \log g_i )$$

Differentiating  
 Now taking ~~log~~ on both side & considering  $g_i$  as constant

$$\delta (\log W) \approx \sum_{i=1}^K ( n_i \delta \log (n_i + g_i) + \log (n_i + g_i) \delta n_i + g_i \delta \log (n_i + g_i) - ( n_i \delta \log n_i + \log n_i \delta n_i ) + 0 )$$

$$\Rightarrow 0 = \sum_{i=1}^K ( n_i \delta \log (n_i + g_i) + g_i \delta \log (n_i + g_i) + \log (n_i + g_i) \delta n_i - n_i \delta \log n_i + \log n_i \delta n_i )$$

$$\Rightarrow 0 = \sum_{i=1}^k \left( (n_i + g_i) \int \log(n_i + g_i) + \log(n_i + g_i) S_{ni} - n_i \int \log n_i - \log n_i S_{ni} \right)$$

$$\Rightarrow 0 = \sum_{i=1}^k \left( \cancel{(n_i + g_i)} \times \frac{1}{\cancel{(n_i + g_i)}} S_{ni} + S_{ni} \log(n_i + g_i) - \cancel{n_i} \times \frac{1}{\cancel{n_i}} S_{ni} - S_{ni} \log n_i \right)$$

$$\Rightarrow 0 = \sum_{i=1}^k \left( S_{ni} \log(n_i + g_i) - S_{ni} \log n_i \right)$$

$$\Rightarrow - \sum_{i=1}^k \left( \log n_i - \log(n_i + g_i) \right) S_{ni} = 0$$

$$\Rightarrow \sum_{i=1}^k \log \left( \frac{n_i}{n_i + g_i} \right) S_{ni} = 0 \quad \text{--- (4)}$$

Now multiplying (1) by  $\alpha$  & (2) by  $\beta$   
Then adding (1) (2) + (4)

$$\Rightarrow \sum_{i=1}^k \left( \log \left( \frac{n_i}{n_i + g_i} \right) + \alpha S_{ni} + \beta E_i S_{ni} \right) = 0$$

$$\Rightarrow \log \left( \frac{n_i}{n_i + g_i} \right) + \alpha + \beta E_i = 0$$

$$\log \left( \frac{n_i}{n_i + g_i} \right) = -\alpha - \beta E_i$$

$$\left( \frac{n_i}{n_i + g_i} \right) = e^{-\alpha - \beta E_i} = e^{-(\alpha + \beta E_i)}$$

$$\frac{n_i + g_i}{n_i} = e^{(\alpha + \beta E_i)}$$

$$1 + \frac{g_i}{n_i} = e^{(\alpha + \beta E_i)}$$

$$\Rightarrow \boxed{n_i = \frac{g_i}{e^{\alpha + \beta E_i} - 1}}$$

## Fermi Dirac Statistics:

Fermi Dirac describe distribution of particles in a system. (identical indistinguishable particles) that obey Pauli Exclusion Principle.

### Basic Postulates:-

- (i) All the particles which obey Fermi-Dirac statistics known as Fermions.
- ii) Fermions obey Uncertainty Principle.
- iii) Fermions obey Pauli Exclusion Principle.
- iv) Fermi Dirac distribution law is represented by -

$$n_i = \frac{g_i}{e^{\epsilon_i/kT} + 1}$$

Let us consider a system of  $N$  independent and indistinguishable particles. Let the compartment be marked  $1, 2, \dots, k$  and their mean energy values be represented by  $E_1, E_2, \dots, E_k$ . It contains  $g_1, g_2, \dots, g_k$  cells. Total number of particles in the system is  $N = n_1 + n_2 + \dots + n_k$ .

Now these particles obey fundamental postulates of quantum statistics. as -

① Total number of particles is constant

$$N = n_1 + n_2 + \dots + n_k = \text{constant}$$

$$\delta N = \delta n_1 + \delta n_2 + \dots + \delta n_k = \text{constant} = 0$$

$$\Rightarrow \left( \sum \delta n_i = 0 \right) \quad \text{--- (1)}$$

② Total Energy of system is constant

$$E = E_1 n_1 + E_2 n_2 + \dots + E_k n_k = \text{constant}$$

$$\delta E = E_1 \delta n_1 + E_2 \delta n_2 + \dots + E_k \delta n_k = 0$$

$$\Rightarrow \left( \sum E_i \delta n_i = 0 \right) \quad \text{--- (2)}$$

③ Distribution of particle (Thermodynamic Probability) is constant

$$\text{i.e. } \frac{\delta W_2}{\int \log W_2} \quad \text{--- (3)}$$

∴ Hence the Number of ways in which  $n_i$  particles, can be distributed in  $g_i$  cells in  $i$ th compartment is -

$$W = \sum_{i=1}^k \frac{g_i!}{n_i! (g_i - n_i)!}$$

Taking log on both sides -

$$\log W = \sum_{i=1}^k \left( \log g_i! - \log n_i! - \log (g_i - n_i)! \right)$$

( Using Stirling Approximation,  $\log N! = N \log N - N$  )

$$\log W = \sum_{i=1}^k \left( g_i \log g_i - g_i - (n_i \log n_i - n_i) - ( (g_i - n_i) \log (g_i - n_i) - (g_i - n_i) ) \right)$$

$$\Rightarrow \log W = \sum_{i=1}^k \left( g_i \log g_i - g_i - n_i \log n_i + n_i - (g_i - n_i) \log (g_i - n_i) + g_i - n_i \right)$$

$$\Rightarrow \log W = \sum_{i=1}^k \left( g_i \log g_i - n_i \log n_i - (g_i - n_i) \log (g_i - n_i) \right)$$

( Now Differentiating both sides and considering  $g_i$  as constant as  $g_i$  is Number of cells which is constant )

$$\Rightarrow \frac{d}{dn_i} \log W = \sum_{i=1}^k \left( 0 - (n_i \log n_i + \log n_i \cdot n_i) - ( (g_i - n_i) \log (g_i - n_i) + \log (g_i - n_i) \cdot (-n_i) ) \right)$$

$$0 = \sum_{i=1}^k \left( 0 - \left( n_i \frac{1}{n_i} n_i + \log n_i \cdot n_i \right) - \left( (g_i - n_i) \frac{1}{(g_i - n_i)} (-n_i) + \log (g_i - n_i) (-n_i) \right) \right)$$

$$0 = \sum_{i=1}^k \left( 0 - 0 - \log n_i \cdot n_i - 0 + \log (g_i - n_i) \cdot n_i \right)$$

$$0 = \sum_{i=1}^k \left( -\log (g_i - n_i) \cdot n_i + \log n_i \cdot n_i \right) \quad \text{--- (4)}$$

∴ Multiplying (1) by  $\alpha$  (2) by  $\beta$  and Adding (1) + (2) + (4)

$$\Rightarrow \sum_{i=1}^k \left( \log \left( \frac{n_i}{g_i - n_i} \right) + \alpha \delta n_i + \beta \delta E_i \right) = 0$$

$$\Rightarrow \sum_{i=1}^k \left( \log \left( \frac{n_i}{g_i - n_i} \right) + \alpha \delta n_i + \beta \delta E_i \right) \delta n_i = 0$$

$$\Rightarrow \log \frac{n_i}{g_i - n_i} = -\alpha - \beta E_i$$

$$\frac{n_i}{g_i - n_i} = e^{-\alpha - \beta E_i} = e^{-(\alpha + \beta E_i)}$$

$$\frac{g_i - n_i}{n_i} = e^{\alpha + \beta E_i}$$

$$\frac{g_i}{n_i} - 1 = e^{\alpha + \beta E_i}$$

$$\frac{g_i}{n_i} = e^{\alpha + \beta E_i} + 1$$

$n_i$	$\frac{g_i}{e^{\alpha + \beta E_i} + 1}$
-------	--

## Applications of Quantum Statistics:-

i) Maxwell Boltzmann (M-B) Statistics:-

Application:- Molecular speed and Energies of an ideal gas.

ii) Bose-Einstein (B-E) Statistics:-

Application - Black body spectrum.

iii) Fermi-Dirac (F-D) Statistics).

Application:- Free Electron Theory.

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## Application of Maxwell's Boltzmann Statistics:-

### (a) Molecular Speed and Energy of an ideal gas.

Let us consider an ideal gas in a container of volume  $V$   
then according to Maxwell Boltzmann Distribution law  
The number of molecule in the  $i^{\text{th}}$  cell of Energy  $E_i$  will be

$$n_i = g_i e^{-(\alpha + \beta E_i)}$$

$$n_i = \frac{g_i}{e^{(\alpha + \beta E_i)}}$$

Substitute  $\beta = \frac{1}{k_B T}$

$$\Rightarrow n_i = \frac{g_i}{e^{(\alpha + E_i/k_B T)}} = g_i e^{-\alpha} e^{-E_i/k_B T}$$

Considering continuous distribution of molecular energy

$$n(E) d(E) = g(E) e^{-\alpha} e^{-(E/k_B T)} dE \quad \text{--- (1)}$$

Now a molecule of Energy  $E$  has momentum  $p$   
whose magnitude is

$$p = \sqrt{2mE}$$

Let us consider a volume of spherical cell in a momentum space of radius  $p$  and thickness  $dp$ . Volume of this spherical cell is

$$4\pi p^2 dp$$

Now  $g(p) dp =$  Number of cell in momentum range  $p$  to  $p+dp$

$$= 4\pi p^2 dp$$

$$= B p^2 dp$$

where  $B$  is some constant.

Since each magnitude of momentum  $p$  correspond to Energy  $E$

Now  $g(E) d(E) =$  Number of Energy State

$$= g(p) dp$$
$$= B p^2 dp$$



But  $p^2 = 2mE$  and  $dp = \frac{2m dE}{\sqrt{2mE}}$

We have

$$\begin{aligned} g(E) d(E) &= B p^2 dp \\ &= B \times \frac{2mE}{\sqrt{2mE}} \times \frac{2m dE}{\sqrt{2mE}} \\ &= B \times \sqrt{2mE} \times 2m dE \\ &= B (2m)^{3/2} \times E^{1/2} dE \end{aligned} \quad \text{--- (2)}$$

Substituting (2) in (1)

From (1)  $n(E) d(E) = g(E) e^{-\alpha} e^{-(E/k_B T)} dE$

$$= B (2m)^{3/2} E^{1/2} \times e^{-\alpha} e^{-(E/k_B T)} dE$$

Let  $B (2m)^{3/2} e^{-\alpha} = \text{constant} = C$

$$\Rightarrow n(E) d(E) = C \sqrt{E} e^{-(E/k_B T)} dE \quad \text{--- (3)}$$

Now to find C make the condition that total No. of molecules is N.

$$N = \int_0^{\infty} n(E) dE = C \int_0^{\infty} \sqrt{E} e^{-(E/k_B T)} dE$$

Using  $\int_0^{\infty} \sqrt{x} e^{-ax} dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}}$

$$\Rightarrow N = C \frac{1}{2(1/k_B T)} \sqrt{\frac{\pi}{(1/k_B T)}} = C \times \frac{\sqrt{\pi}}{2} \times (k_B T)^{3/2}$$

$$\Rightarrow C = \frac{2\pi N}{(\pi k_B T)^{3/2}}$$

Put value of C in (3)

$$\Rightarrow n(E) d(E) = \frac{2\pi N}{(\pi k_B T)^{3/2}} \times \sqrt{E} e^{-(E/k_B T)} dE \quad \text{--- (4)}$$

\* Now total Energy of the system is -

$$E = \int_0^{\infty} E n(E) dE = \frac{2\pi N}{(\pi k_B T)^{3/2}} \int_0^{\infty} E^{3/2} e^{-E/k_B T} dE$$

$$= \frac{2\pi N}{(\pi k_B T)^{3/2}} \times \frac{3}{4} (k_B T)^2 \sqrt{\pi k_B T}$$

( Using  $\int_0^{\infty} \sqrt{x} e^{-ax} dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}}$  )

$$E = \frac{3}{2} k_B T$$

b) Maxwell Boltzmann velocity distribution.

Now Number of Molecule with velocity  $v$  &  $v + dv$  is,

Substituting  $E = \frac{1}{2} m v^2$ ,  $dE = m v dv$ , in (4)

we get

$$n(v) dv = \frac{(\sqrt{2} \pi N m^{3/2})}{(\pi k_B T)^{3/2}} v^2 e^{-mv^2/2k_B T} dv$$

## Some Molecular Speeds!:-

- 1) Mean or Average Speed!:- Average speed of all the molecules.

$$V_{avg} = \sqrt{\frac{8KT}{\pi m}}$$

- 2) Root Mean Square Speed!:- Mean speed of square of all the molecule.

$$V_{rms} = \sqrt{\frac{3KT}{m}}$$

- 3) Most Probable Speed!:- Speed possessed by maximum number of molecule.

$$V_{mp} = \sqrt{\frac{2KT}{m}}$$

Relation between  $V_{rms}$ ,  $V_{mp}$  &  $V_{avg}$ .

$$\begin{aligned} V_{rms} : V_{avg} : V_{mp} &= \sqrt{\frac{3KT}{m}} : \sqrt{\frac{8KT}{\pi m}} : \sqrt{\frac{2KT}{m}} \\ &= 1.73 \sqrt{\frac{KT}{m}} : 1.59 \sqrt{\frac{KT}{m}} : 1.414 \sqrt{\frac{KT}{m}} \\ &= 1 : 0.92 : 0.817 \end{aligned}$$

$$V_{rms} > V_{avg} > V_{mp}$$

## Applications of Fermi Dirac Statistics:-

### Free Electron Theory:-

It is of 2 type

- I) Classical (Lorentz-Drude Theory)
- II) Quantum (Sommerfeld Theory).

### Classical Free Electron Theory:

According to classical free electron theory electrons are assumed completely free. A single free electron is surrounded by large number of free electron and positive ion core.

#### ASSUMPTIONS:

- \* In this theory it is assumed that when a free electron moves in a solid its potential energy is zero. i.e. electron can move freely throughout the crystal.
- \* Electron can move freely inside the solid, but it can't leave the surface. (outside no force on electron but inside a constant force on electron). This is an infinite barrier so no electron can cross the barrier.



Features:- Free electron gas obey Maxwell Boltzmann statistics.

- i) It successfully defines Ohm's law.
- ii) These free electrons can move freely so metals have high electrical & thermal conductivity.

## Failure of Classical Free electron theory.

- I) It can't explain temperature dependence of Electrical Resistivity.
- II) It fails to explain Electronic heat capacity and Paramagnetic susceptibility of Free electron.

## Quantum Free Electron theory:

Classical theory assumed that electron in Free Electron gas follow M-B Statistics. but in Quantum Free Electron theory electron follow Fermi Dirac Statistics as Pauli Exclusion Principle applicable on Electron.

→ All Assumption of Quantum Free Electron theory are same as classical Free Electron theory except now Electron follow Fermi Dirac Statistics. (Pauli Exclusion Principle).

Note:- (Pauli Exclusion Principle:- No two Electron can have same Quantum State).

Assumptions

- I) Energy of Free electron are quantized
- II) Free  $e^-$  obey Pauli Exclusion Principle
- III) Distribution of Free  $e^-$  obey Fermi-Dirac quantum Statistics.
- IV) Free  $e^-$  have same potential Energy everywhere within metal
- V) Electrons are treated free

## Energy Distribution among free electron!

We have to find distribution of Electron Energy in a metal

Number of Electron with Energy  $E$  is given by -

$$n(E) d(E) = \frac{g(E) d(E)}{e^{\lambda} e^{E/KT} + 1} \quad \text{--- (1)}$$

Electron Energy State is given by -

$$g(E) d(E) = \frac{8\sqrt{2} \pi V m^{3/2} E^{1/2} dE}{h^3}$$

Substituting  $g(E) d(E)$  in (1)

$$n(E) d(E) = \frac{8\sqrt{2} \pi V m^{3/2} E^{1/2} dE}{h^3} \frac{1}{e^{\lambda} e^{E/KT} + 1}$$

$$\lambda = \frac{E_F}{KT}$$

$$n(E) d(E) = \frac{8\sqrt{2} \pi V m^{3/2} E^{1/2} dE}{h^3} \frac{1}{e^{(E-E_F)/KT} + 1}$$

Fermi Energy: Fermi Energy is the Energy difference between highest & lowest occupied state.

Suppose a metal contains  $N$  free Electrons

Total Number of Free Electrons is equal to the total number of quantum states upto Energy  $E_F$ .

$$N = \int_0^{E_F} g(E) dE$$

$$N = \frac{8\sqrt{2}\pi V m^{3/2}}{h^3} \int_0^{E_F} E^{1/2} dE$$

$$= \frac{16\sqrt{2}\pi V m^{3/2}}{3h^3} E_F^{3/2}$$

$$= \frac{8\pi V}{3} \left(\frac{2m}{h^2}\right)^{3/2} E_F^{3/2}$$

$$E_F = \frac{h^2}{2m} \left(\frac{3N}{8\pi V}\right)^{2/3}$$

— Fermi Energy

using  
( $k = \frac{h}{2\pi}$ )

$$E_F = \frac{h^2}{2m} \left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

## Electronic Specific Heat:-

Electronic Specific Heat of Electron is defined as the Ratio of portion of Heat used by Electron to Rise the temperature of the system.

we have

Internal Energy Per mole of metal is.

$$E_e = \frac{3}{2} NkT = \frac{3}{2} RT \quad \left( k = \frac{R}{N} \right)$$

$R$  - Gas Constant.  $N$  - ~~Force~~ Avogadro Number.

$N$  - Avogadro Number.

Then molar Specific Heat of Electron should be.

$$C_{ve} = \left( \frac{dE_e}{dT} \right)_v = \frac{3}{2} R$$

Total specific heat of metal should be

$$C_v = 3R + \frac{3}{2} R = \frac{9}{2} R$$

$$\boxed{C_{ve} = \frac{3}{2} R}$$

- Electronic Specific Heat.



## Fermi Distribution Function:

$$f(E) = \frac{1}{e^{\left(\frac{E-E_F}{kT}\right)} + 1}$$

This describes the probability of occupancy of  $N e^-$ .  
OR

At temperature  $T$  the probability of occupation of an electron state of energy  $E$  is given by Fermi Distribution Function.

### Behaviour at Different temperature:

i)  $T=0K$ ,  $E < E_F$

$$f(E) = \frac{1}{e^{-\infty} + 1} = \frac{1}{e^{-\infty} + 1} \approx 1$$

$$\Rightarrow \boxed{f(E) = 1}$$

Probability of state to be filled is 1 i.e. state is 100% filled.

ii)  $T=0K$ ,  $E > E_F$

$$f(E) = \frac{1}{e^{+\infty} + 1} = \frac{1}{e^{+\infty} + 1} = 0$$

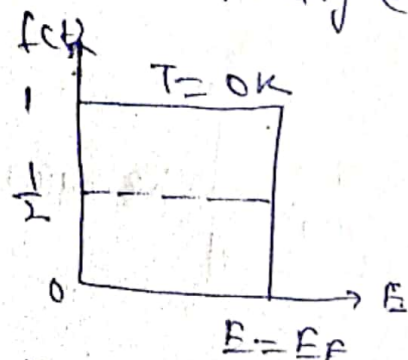
$$\Rightarrow \boxed{f(E) = 0}$$

It means no state is filled i.e. completely empty.

iii)  $T=0$ ,  $E = E_F$

$$\boxed{f(E) = \frac{1}{2}}$$

Fermi level is the level of 50% occupancy. Fermi level lying between completely filled and completely empty level.



Q. Use Fermi distribution function to obtain value  $F(E)$  for  $(E - E_F) = 0.01 \text{ eV}$  at  $200 \text{ K}$ .

Sol. Difference between Energy level  $(E - E_F) = 0.01 \text{ eV}$   
 $= 0.01 \times 1.6 \times 10^{-19} \text{ J} = 1.6 \times 10^{-21} \text{ J}$

We have Fermi Distribution function -

$$F(E) = \frac{1}{1 + e^{(E - E_F)/kT}}$$
$$= \frac{1}{1 + e^{(1.6 \times 10^{-21}) / (1.38 \times 10^{-23} \times 200)}}$$
$$= \frac{1}{1 + e^{0.5797}} = 0.3589$$

Q. Calculate Fermi Energy & Fermi temperature in a metal. Fermi velocity of Electron in Metal is  $0.86 \times 10^6 \text{ ms}^{-1}$

Sol. Given  
velocity of electron  $v = 0.86 \times 10^6 \text{ ms}^{-1}$   
Mass of Electron  $m = 9.1 \times 10^{-31} \text{ kg}$   
Electronic charge  $e = 1.6 \times 10^{-19} \text{ C}$   
Boltzmann's constant  $k = 1.38 \times 10^{-23} \text{ J K}^{-1}$

Now

$$\text{Fermi Energy } E_F = \frac{1}{2} m v^2$$
$$= \frac{1}{2} \times 9.1 \times 10^{-31} \times (0.86 \times 10^6)^2$$
$$= 3.368 \times 10^{-19} \text{ J}$$

$$\text{Fermi Temperature } T_F = \frac{E_F}{k}$$
$$= \frac{3.368 \times 10^{-19}}{1.38 \times 10^{-23}} = 2.43 \times 10^4 \text{ K}$$

Fermi Temperature  $T_F$  is  $2.43 \times 10^4 \text{ K}$

## Numericals! -

Q Calculate the different number of ways of arranging 6 fermions in 6 elementary phase cells.

Sol. According to Fermi-Dirac statistics different ways of arranging  $n_i$  fermions in  $g_i$  phase cell of  $i$ th energy level is given by -

$$(W)_{F-D} = \frac{g_i!}{(g_i - n_i)! n_i!}$$

Hence  $n_i = 6$  &  $g_i = 6$

$$\begin{aligned} &= \frac{6!}{(6-6)! 6!} \\ &= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 1 \end{aligned}$$

Q Calculate the number of different arrangement of 8 indistinguishable particles among seven cells, there being no restriction on number of particles in obeying B-E statistics.

Sol. According to B-E statistics no. of different arrangement of  $n_i$  indistinguishable particle in  $g_i$  cell is given by -

$$(W)_{B-E} = \frac{(n_i + g_i - 1)!}{n_i! (g_i - 1)!}$$

$n_i = 8$ ,  $g_i = 7$

Hence  $(W)_{B-E} = \frac{(8+7-1)!}{8! \times (7-1)!} = \frac{14!}{8! 6!}$

$$= \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8!}{8! \times (6 \times 5 \times 4 \times 3 \times 2 \times 1)}$$

$$= 3003$$

**Example 3.1** An electron gas obeys the Maxwell-Boltzman statistics. Calculate average thermal energy (in eV) of an electron in the system at 300 K. [GGSIPU, March 2015 (2 marks)]

**Solution.**  $E = \frac{3}{2} k_B T = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 \text{ J} = \frac{3 \times (1.38 \times 10^{-23}) \times 300}{2 \pi (1.6 \times 10^{-19})} \text{ eV} = 0.039 \text{ eV}$

**Example 3.2** At what temperature would one in a thousand of atom in a gas of atom hydrogen be in  $n=2$  energy level ?

**Solution.** For hydrogen  $n(E) = -\frac{13.6}{n^2} \text{ eV}$   
 $n(E_1) = n_1 = -13.6 \text{ eV}$   
 $n(E_2) = n_2 = -3.4 \text{ eV}$

$g(E) = \text{no. of states formed} = 2n^2$

$g(E_1) = g_1 = 2$  and  $g(E_2) = g_2 = 8$

For Maxwell-Boltzmann distribution is

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \frac{e^{-E_2/k_B T}}{e^{-E_1/k_B T}} = \frac{8}{2} e^{-(E_2 - E_1)/k_B T}$$

or  $\frac{1}{10^3} = 4 e^{-(E_2 - E_1)/k_B T}$  or  $e^{(E_2 - E_1)/k_B T} = \frac{10^4}{2.5} = 4000$

Taking logarithms both the sides

$$\frac{E_2 - E_1}{k_B T} = \ln 4000$$

$$k_B T = \frac{(E_2 - E_1)}{\ln 4000} = \frac{10.2 \text{ eV}}{8.29}$$

$$T = \frac{10.2 \times 1.6 \times 10^{-19}}{8.29 \times 1.38 \times 10^{-23}} \text{ K} = 1.43 \times 10^4 \text{ K}$$

$$T = 14300 \text{ K} = 14300 - 273 = 14027^\circ \text{C}$$

## Dying Stars:-

### 1) White / Black Dwarfs.

A star like our sun will become a white dwarf when it will exhaust its nuclear fuel; which then settle down to become a white dwarf.

White dwarfs have no way to keep themselves hot, eventually such stars cool completely & become black dwarfs.

Typical data of white Dwarfs -

Content - mostly Helium

Density -  $10^7 \text{ g/cc}$

mass -  $10^{33} \text{ g} = \text{Mass of Sun}$

Central Temperature =  $10^7 \text{ K} = \text{Sun's Temperature}$ .

At its central temperature Helium atoms are expected to be completely ionised, we regard the gas of electron as ideal Fermi gas.

$$E_F = \frac{\hbar^2}{2m} \left( \frac{3N}{8\pi V} \right)^{3/2} = 20 \text{ MeV}$$

$$T_F = 10^{11} \text{ K}$$

Also No white ~~star~~ dwarf star can have mass larger than

$$M_0 = 10^{23} \text{ g} \approx \text{Mass of Sun}$$

If,  $M_0 = 1.4 \times \text{Mass of Sun}$

This mass is known as Chandrasekhar limit.

## ii) Neutron Stars:-

Neutron stars are typically about 10 miles in diameter have about 1.4 time the mass of a sun and spin very rapidly. Neutron stars are fascinating because they are densest object known. Due to small size and high density, a neutron star posses a surface gravitational field of about 300,000 times that of Earth.

Magnetic field about 1,000,000,000,000 times stronger than Earth.

## iii) Black holes:-

Black holes are object so dense that not even light can escape their gravity and nothing can escape from inside a black hole.

Two Existence of two types of black holes:

i) Mass with 4-5 times of mass of sun.

ii) with mass of a typical galaxy.

This evidence comes not from seeing the black holes directly, but by observing the behaviour of stars & other material near them.

## Formulae at a Glance

### 3.1 Volume of each phase cell

$$\delta V = \delta x \delta y \delta z \delta p_x \delta p_y \delta p_z$$

$$\delta x \delta p_x \geq \hbar, \delta y \delta p_y \geq \hbar \text{ and } \delta z \delta p_z \geq \hbar$$

$$\delta V = \hbar^3 \quad \left[ \because \hbar = \frac{h}{2\pi} \right]$$

### 3.2 Maxwell-Boltzmann Statistics

(i) The distribution law  $n_i = \frac{g_i}{e^{(\alpha + E_i/k_B T)}}$

(ii) Molecular energy in an ideal gas

$$E = \frac{2\pi N}{(\pi k_B T)^{3/2}} \frac{3}{4} (k_B T)^2 \sqrt{\pi k_B T} = \frac{3}{2} N k_B T$$

and  $\bar{E} = \frac{3}{2} k_B T$

(iii) Maxwell-Boltzmann velocity distribution law

$$n(v)dv = \frac{\sqrt{2} N m^{3/2}}{(\pi k_B T)^{3/2}} v^2 e^{-\frac{mv^2}{2k_B T}} dv$$

(a) RMS velocity  $v_{rms} = \sqrt{\frac{3k_B T}{m}}$

(b) Average velocity  $= \bar{v} = \sqrt{\frac{8k_B T}{\pi m}}$

(c) Most probable velocity  $(v_{mp}) = \sqrt{\frac{2k_B T}{m}}$

3.3 (i) Symmetric wave function  $\psi_s(1,2) = +\psi(2,1)$

(ii) Antisymmetric wave function

$$\psi_A(1,2) = -\psi(2,1)$$

### 3.4 Bose-Einstein statistics

(i) Distribution law  $n_i = \frac{g_i}{(e^{\alpha + E_i + k_B T} - 1)}$

(ii) Planck's radiation law

$$dE = E(v)dv = \frac{8\pi h v^3}{c^3} \frac{1}{(e^{hv/k_B T} - 1)} dv$$

(iii) Rayleigh-Jeans law  $E(v)dv = \frac{8\pi v^2 k_B T}{c^3} dv$

(iv) Wein's displacement law

$$\lambda_{max} T = \text{Constant}$$

(v) Bose-Einstein Condensation

(a) Critical temperature

$$T_C = \frac{h^3}{2\pi m k_B T} \left( \frac{1}{2612} \frac{n}{V} \right)^{2/3}$$

(b) Rest of particles

$$n_0 = n \left[ 1 - \Sigma \left( \frac{T}{T_C} \right)^{3/2} \right] \text{ for } T < T_C$$

### 3.5 Fermi Dirac Statistics

(i) Distribution law  $n_i = \frac{g_i}{(e^{\alpha + E_i/k_B T} + 1)}$

(ii) Fermi momentum  $p_F = \left( \frac{3N}{8\pi V} \right)^{1/3} h$

(iii) Fermi Energy  $E_F = \frac{p_F^2}{2m} = \frac{h^2}{2m} \left( \frac{3N}{8\pi V} \right)^{2/3}$

(iv) Ground state energy

$$E_0 = \frac{8\pi V}{h^3} \frac{p_F^5}{10m} = \frac{3}{5} N E_F$$

(v) Mean energy of Fermion  $\bar{E} = \frac{E_0}{N} = \frac{3}{5} E_F$

(vi) Electronic heat capacity

$$(C_v)_{El} = \frac{\pi^2}{2} N k_B \left( \frac{T}{T_F} \right) = aT$$

(vii) Sommerfield constant  $a = \frac{N k_B^2 \pi^2}{2 E_F}$

## Miscellaneous Solved Numerical Problems

**Problem 3.1** A gas has only two particles A and B Show that with the help of diagrams, how these two particles can be arranged in three series 1, 2, 3 using (i) Maxwell-Boltzmann, (ii) Bose-Einstein and (iii) Fermi-Dirac statistics.

**Solution. (i) Maxwell-Boltzmann statistics :**

The two particles are distinguishable.

There is no limit to the number of particles in any one state.

The total number of ways =  $3^2 = 9$ .

States	Possible distribution in various states								
1	A	B	-	-	A	B	AB	-	-
2	B	A	A	B	-	-	-	AB	-
3	-	-	B	A	B	A	-	-	AB

**(ii) Bose-Einstein statistics :**

If A and B are quantum particles, they are indistinguishable. Thus they have to be given the same name, say A.

There is no limit to the number of particles in any one state.

The total number of ways = 6

States	Possible distribution in various states					
1	A	A	-	AA	-	-
2	A	-	A	-	AA	-
3	-	A	A	-	-	AA

**(iii) Fermi-Dirac statistics :**

The particles are indistinguishable and not more than one particle can be in any one state.

The total number of ways = 3

States	Possible distribution in various states		
1	A	A	-
2	A	-	A
3	-	A	A

**Problem 3.2** Show that on increasing temperature, the number of atoms in excited state increases.

**Solution.**  $n(E) = g(E) \cdot f(E)$

For Maxwell-Boltzmann distribution

$$n(E_i) = \frac{g(E_i)}{e^{(\alpha + E_i/k_B T)}} = g(E_i) A e^{-E_i/k_B T}$$

Then  $n(E_1) = g(E_1) A e^{-E_1/k_B T}$

and  $n(E_2) = g(E_2) A e^{-E_2/k_B T}$

$\Rightarrow \frac{n(E_2)}{n(E_1)} = \frac{g(E_2)}{g(E_1)} e^{-(E_2 - E_1)/k_B T}$

On increasing temperature  $\Rightarrow n(E_2) > n(E_1)$



**Problem 3.3** (a) Show that at  $E = E_F$  and  $T \neq 0K$ . The occupancy is equal to 0.5. (b) Show that at  $E < E_F$  and  $T = 0K$ . The occupancy of state is perpendicular and at  $E > E_F$  and  $T = 0K$ , the occupancy of state is zero.

**Solution.** From Fermi-Dirac statistics

$$n_i = \frac{g_i}{e^{\alpha + E_i/k_B T} + 1}$$

This is written as 
$$n(E) = \frac{g(E)}{e^{(E-E_F)/k_B T} + 1} = g(E) f(E)$$

Then occupancy 
$$f(E) = \frac{1}{(e^{(E-E_F)/k_B T} + 1)}$$

(a) When  $T \neq 0$  and  $E = E_F$  then 
$$f(E) = \frac{1}{e^0 + 1} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

(b) When  $T = 0$  and  $E < E_F$  then 
$$f(E) = \frac{1}{e^{-\infty} + 1} = \frac{1}{0+1} = 1$$

**Problem 3.7** At what temperature can we expect a 10% probability that electrons in a metal will have an energy which is 1% above  $E_F$ ? The Fermi energy of the metal is 5.5 eV. [IGGSIPU, May 2014 (4.5 marks)]

**Solution.** Given :  $f(E) = 10\%$ ,  $E = E_F + 1\%$  of  $E_F$ ,  $E_F = 5.5$  eV,  $T = ?$

$$f(E) = \frac{1}{e^{(E-E_F)/k_B T} + 1}$$

$$E = 5.5 + \frac{5.5}{100} = 5.5 + 0.555; \quad E - E_F = 0.555.$$

$$0.1 = \frac{1}{\left( \exp. \frac{0.555 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23} T} \right) + 1} = \frac{1}{\exp. \left( \frac{637.7}{T} \right) + 1} \quad \text{or} \quad T = 290.2 \text{ K}$$